ON THE INTRODUCTION OF A RECYCLING ACTIVITY INTO AN INPUT-OUTPUT SYSTEM

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ABSTRACT In this paper a recycling activity is introduced into an input-output model in which ordinary production as well as pollutant abatement activities take place, and the effects of theintroduction of the recycling activity are examined. The recycled good is assumed to be different from the other commodities. Similarly, the pollutant used in the recycling activity either has or has not been subject to the abatement. Three different combinations of activity form the analytical framework, For all these combinations, it is the case that if the recycling activity is efficient the introduction of recycling has favourable effects upon employment as well as the environmental aspects, although the criterion for the efficiency is slightly different from case to case.

1. INTRODUCTION

Nearly three decades have elapsed since Leontief (1970) published a pioneering article that applied input-output analysis to environmental problems. However, it seems that little attention has been paid to the analysis of recycling activities within the framework of input-output analysis. A probable reason for this is that recycling activities have to be treated by assuming joint production and not the usual non-joint (or single production) which is an implicit postulate of input-output analysis. This is because the pollutants discharged from production processes are reckoned as the by-products of those processes. An explicit recognition of joint production often leads to analytical difficulties and hence yields few satisfactory results as in the case of von Neumann models. Therefore, it would be of some value to investigate the environmental repercussions of recycling activities within the framework of input-output analysis.

To this end, we first consider an economy where r-1 useful goods (henceforth, goods or good, for simplicity) labelled 1 through r-1, are produced with the emission of n-r pollutants labelled r+1 through n. If necessary the number n-r of pollutants is signified by s. The index r denotes the recycled good exclusively, unless otherwise specified. Following Leontief (1970), we assume that the pollutants in this economy are "abated", (meaning that the pollutant is, at least partly, made harmless).

Section 2 defines the notation used, notes a preliminary consideration and enumerates the three cases that will be studied in the subsequent section. In Section 3 we present our basic model and argue the cases one by one. Section 4 offers concluding remarks and highlights any problems that remain to be studied further.

2. NOTATION AND PRELIMINARY MATTERS

The notation and symbols used in the analysis that follows are:

A_{11} = $[a_{ij}]$; $i = 1, \cdots, r-1$; $j = 1, \cdots, r-1$ where a_{ij} is the input of good i per unit of output of good j (produced by sector j).
A_{12} = $[a_{ij}]$; $i = 1, \cdots, r-1$; $j = r+1, \cdots, n\ (=r+s)$ where a_{ij} is the input of good i per unit of abated pollutant j (abated by sector j).
A_{21} = $[a_{ij}]$; $i = r+1, \cdots, n$; $j = 1, \cdots, r-1$ where a_{ij} is the output of pollutant i per unit of output of good j.
A_{22} = $[a_{ij}]$; $i = r+1, \cdots, n$; $j = r+1, \cdots, n$ where a_{ij} is the output of pollutant i per unit of abated pollutant j (abated by sector j).

The technical coefficients to represent the recycling activity assumed to convert a pollutant, say n, into a useful recycled good r are as follows:
a_{rj} ; $j = 1, \cdots, n$ is the input of recycled good r per unit of output j $(j = 1, \cdots, r)$ or of abated pollutant j $(j = r+1, \cdots, n)$.
a_{ir} ; $i = 1, \cdots, n-1$ is the input of good i $(i = 1, \cdots, r)$ or output of pollutant i $(i = r+1, \cdots, n-1)$ per unit of output of recycled good.
a_{nr} is the input of pollutant n per unit of output of recycled good r, hence a_{nr} is negative.
In vector notation, these coefficients are written as $a_{r1} = [a_{r1}, \cdots, a_{r\,r-1}]$, $a_{r2} = [a_{r\,r+1}, \cdots, a_{rn}]$, $a_1^r = [a_{1r}, \cdots, a_{r-1\,r}]'$ and $a_2^r = [a_{r+1\,r}, \cdots, a_{nr}]'$, where $'$ denotes the transposition of a vector or matrix to which it is attached.

$x_1 = [x_1, \cdots, x_{r-1}]'$ is the vector of output of good j $(j = 1, \cdots, r-1)$.
$x_2 = [x_{r+1}, \cdots, x_n]'$ is the vector of output of abated pollutant j $(j = r+1, \cdots, n)$.
$c = [c_1, \cdots, c_r]'$ is the vector of final demands for good j $(j = 1, \cdots, r)$.
$u = [u_{r+1}, \cdots, u_n]'$ is the vector of upper bounds on the pollutant j $(j = r+1, \cdots, n)$ which remaines unabated.

We now list the matrices of technical coefficients before and after the introduction of the recycling activity under investigation.

$$A = \begin{bmatrix} A_{11} & 0 & A_{12} \\ 0' & 0 & 0' \\ A_{21} & 0 & A_{22} \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} A_{11} & a_1^r & A_{12} \\ a_{r1} & a_{rr} & a_{r2} \\ A_{21} & a_2^r & A_{22} \end{bmatrix} \quad \hat{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad D = \begin{bmatrix} 0 & a_1^r & 0 \\ a_{r1} & a_{rr} & a_{r2} \\ 0 & a_2^r & 0 \end{bmatrix}$$

where 0 is a column vector consisting of the appropriate number of zeros.

The inverses of $(I - A)$, $(I - \tilde{A})$ and $(I - \hat{A})$, if they exist, are presented by B, $\tilde{B}$ and $\hat{B}$ respectively. In addition, B and $\tilde{B}$ are, if necessary, partitioned in the same manner as A and $\tilde{A}$ have been. Thus, we have:

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} & \tilde{B}_{13} \\ \tilde{B}_{21} & \tilde{B}_{22} & \tilde{B}_{23} \\ \tilde{B}_{31} & \tilde{B}_{32} & \tilde{B}_{33} \end{bmatrix}$$

For later convenience, we write the i-th row (the j-th column) of a matrix, say B, as b_i (b^j) and we denote the i-th row unit vector (the j-th column unit vector) of suitable order by e_i (e^j) Therefore, $[B_{21}\, B_{22}\, B_{23}] = b_r = e_r$, and $[B_{12}'\, B_{22}'\, B_{32}'] = b^r = e^r$. Similarly, $[\tilde{B}_{21}\, \tilde{B}_{22}\, \tilde{B}_{23}] = \tilde{b}_r$ and $[\tilde{B}_{12}'\, \tilde{B}_{22}'\, \tilde{B}_{32}'] = \tilde{b}^r$.

Having dealt with notational matters, let us now proceed to a preliminary consideration - whether the recycled good r is qualitatively different from the other good (represented by D) or not (represented by S) and also whether pollutant n has already been abated (represented by A) or not (represented by NA). All of the logical possibilities are: Case 1 (D) and (A), Case 2 (D) and (NA), Case 3 (S) and (A), and Case 4 (S) and (NA). We will confine ourselves to the analysis of the first three categories, for as will be stated later, Case 4 is dealt with by combining the method used to analyse Case 2 with the aggregation scheme that deals with (S).

This section is finalised by stating our fundamental assumptions and their direct implications. First. the assumption of non-joint production is taken for granted. Second, conventional as it is, we suppose that

(H) $(I - A)$ satisfies the Hawkins-Simon conditions.

Here it should be noted that (H) is equivalent to

(H') $(I - \hat{A})$ is non-negatively invertible.

Since $(I - \hat{A})$ is a principal submatrix of $(I - A)$, (H) obviously implies (H'). Conversely, (H') implies that there exists $x_i > 0$ $(i = 1, 2)$ such that $[I - A]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0$. Let z be any positive number. Then

$$[I - A]\begin{bmatrix} x_1 \\ z \\ x_2 \end{bmatrix} = \begin{bmatrix} (I - A_{11})x_1 - A_{12}x_2 \\ z \\ -A_{21}x_1 + (I - A_{22})x_2 \end{bmatrix} > 0$$

Thus, $(I - A)$ is shown to satisfy the Hawkins-Simon conditions.[1]

The non-singularity of B further establishes the following lemma:

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1 See Kemp and Kimura (1978), p.7.

Lemma 1. Under (H), the set of all eigenvalues of BD coincides with that of DB.

Proof. Let v be any eigenvalue of BD and y be the corresponding row eigenvector. Then, by definition, y is non-zero and $y(BD) = vy$. Postmultiplying both sides of this equality by B, we have $(yB)(DB) = v(yB)$. Noticing that a non-singular matrix transforms a nonzero vector to a nonzero vector, v is an eigenvalue of DB. By w and z denote an eigenvalue of DB and the corresponding column eigenvector. Then from $(DB)z = wz$ and the non-singularity of B coupled with the fact that $z \neq 0$, it follows that $(BD)(Bz) = w(Bz)$ and that $(Bz) \neq 0$. Hence w is an eigenvalue of BD. This completes the proof.

3. ANALYSIS OF RECYCLING

As a starting point, we consider the input-output system before the recycling activity is introduced. Recalling that each final demand cannot exceed the net output (gross output - interindustrial uses) of that good and that every pollutant remaining unabated must not exceed the corresponding upper bound, we have:

$$(I - A)x \leq \begin{bmatrix} c(r) \\ 0 \\ -u \end{bmatrix} \tag{1}$$

where $x = [x_1', 0, x_2']'$ and $c(r)$ is a subvector obtained from c by deleting the r-th element c_r.

Following convention, we will assume that loose inequality (1) contains no strict inequality. Under this assumption we introduce the recycling activity and suppose that Case 1 holds. Then the system to be studied becomes:

$$(I - \tilde{A})\tilde{x} \leq \begin{bmatrix} c \\ 0 \\ -u \end{bmatrix} \tag{2}$$

where $\tilde{x} = [x_1', x_r, x_2']'$.

Our first task is to calculate $\tilde{B}$ if it exists. Since $\tilde{A} = A + D$, $(I - \tilde{A}) = (I - A)(I - BD)$. Hence $(I - \tilde{A})$ is non-singular if and only if this is true and then so is $(I - BD)$. A direct calculation of $(I - BD)$ gives

$$(I - BD) = \begin{bmatrix} I & -(B_{11}a_1^r + B_{13}a_2^r) & 0 \\ a_{r1} & 1 - a_{rr} & -a_{r2} \\ 0 & -(B_{31}a_1^r + B_{33}a_2^r) & I \end{bmatrix}$$

An inspection of the matrix D suggests that the matrix BD is probably non-negative for it is only a_2^r which contains a negative element (a_{nr}). Obviously

$$(eBD)_j = \begin{cases} a_{rj} e b^j & j \neq r \\ eBa^r & j = r \end{cases}$$

where e is a row vector of appropriate order, whose elements are unity (a sum vector). Notice here that there is little possibility of using the recycled good to produce other goods or of abating the pollutant until there is available information about the quality of the recycled good.

Then, it is plausible to expect that $a_{rj} e b^j$ $(j \neq r)$ does not exceed unity and if a_{ir}s $(i = 1, \cdots, n-1)$ are sufficiently small, i.e., if the recycling activity is efficient enough, we can safely assume that $eBa^r < 1$ since $eBa^r < eBa_n^r$, where $a_n^r = [a_{1r}, \cdots, a_{n-1r}, 0]'$. We have thus shown that the supposition that the Frobenius eigenvalue β of BD is less than unity, is not too stringent because β is known not to exceed every column sum of BD. Because Frobenius' theorem on non-negative matrices guarantees that $(I - BD)^{-1}$ exists and is non-negative and Lemma 1 ensures that β is the eigenvalue of DB with the largest absolute value then $(I - DB)$ is surely non-singular and the infinite power series of DB converges to $(I - DB)^{-1}$. This enables us to assert that:

$$\begin{aligned} \tilde{B} &= B(I - DB)^{-1} \\ &= B(I + DB + (DB)^2 + \cdots) \\ &= B(I + D(I + BD + (BD)^2 + \cdots)B) \\ &= B + D(I - BD)^{-1}B \end{aligned}$$

$D(I - BD)^{-1}B = F$ represents the changes in the Leontief inverse due to the newly introduced recycling activity.

To summarise, we can say that in our discussion about recycling, BD, $(I - BD)^{-1}$ and $\tilde{B}$ are non-negative and recycling activity is likely to increase almost all multisectoral multipliers, although there may be a few exceptions.

So far, we have concentrated our attention on the effects of a recycling activity on the Leontief inverse. We now turn to the problem of how to evaluate the recycling activity itself. One strategy is to compare the situation investigated with a situation in which good r is produced without the pollutant n. Let A^* be the matrix obtained from $\tilde{A}$ by replacing a_{nr} with 0 and let E be a matrix whose elements are null except for the (n, r) element which equals $|a_{nr}|$. Then $A^* = \tilde{A} + E$ and therefore $(I - A^*) = (I - \tilde{A} - E)$. Computing the exogenous vector which attains the same outputs that are yielded in the recycled case, we see that the answer is given by

$$\begin{aligned} (I - A^*)(\tilde{B}g) &= (I - E\tilde{B})(I - \tilde{A})(\tilde{B}g) \\ &= (I - E\tilde{B})g \\ &= [c_1, \cdots, c_r, -u_{r+1}, \cdots, -u_{n-1}, -u_n - |a_{nr}|\tilde{b}_r g]' \end{aligned}$$

where $g = [c', -u']'$.

Therefore we can assert that without the abatement of pollutant n due to the operation of the recycling activity, pollutant n would increase by $-|a_{nr}|\tilde{b}_r g$ given the same output.

In the evaluation, we have made no mention of labour markets. However, a fully automated recycling process is unlikely, so it makes sense to examine the effect of the recycling activity on employment.

A thorough study of this effect needs an extended input-output system of considerable complexity, such as Batey's model 10 (1985, p77; originally developed by Oosterhaven (1981)). Nevertheless, for our purposes we can make use of a simple extended input-output system, obtained from the model in Figure 1 of Madden *et al.* (1996. p.210) by discarding the complexities arising from consumers' behaviour and different patterns of income-formation (labour income). This simplification still allows us to study the essence of our problem.

If v_j denotes the labour input coefficients of industry j ($j = 1, \cdots, n$) and v_{I}, v_{II} and v are $(v_1, \cdots, v_{r-1})$, $(v_{r+1}, \cdots, v_n)$ and $(v_{\mathrm{I}}, v_r, v_{\mathrm{II}})$ respectively then, the variation L of the employment caused by the recycling industry is computed to give:

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\begin{aligned}
L &= v\tilde{B}g - [v_{\mathrm{I}}, 0, v_{\mathrm{II}}]B\tilde{y} \\
&= [v_{\mathrm{I}}, 0, v_{\mathrm{II}}]F\tilde{y} + v_r e_r F\tilde{y} + [v_r e_r]F[c_r e^{r}] + v_r c_r \qquad (3) \\
&= [v_{\mathrm{I}} a_1^{r} + v_{\mathrm{II}} a_2^{r}](I - BD)^{-1}(B\tilde{y}) + v_r a_r (I - BD)^{-1}(B\tilde{y}) + v_r c_r f_{rr} + v_r c_r
\end{aligned}
$$

because $\tilde{B} = B + F$, $\tilde{y} = [c(r)', 0, -u']$, $g = \tilde{y} + c_r e^{r}$ and $v = [v_{\mathrm{I}}, 0, v_{\mathrm{II}}] + v_r e_r$.

Equation (3), in particular the second part, implies that the total change in employment is decomposed into:

1. Changes in employment in non-recycling industries. These are caused by the shift in the Leontief inverse due to the fact that the recycling industry starts (hereafter, the shift in the Leontief inverse, for simplicity) with unchanged exogenous factors $(\tilde{y})$.
2. Changes in employment that the recycling industry *would* generate if the exogenous factors were kept intact in spite of the start of recycling activity.
3. The contribution of the recycling industry to the changes in employment due to the shift in the Leontief inverse as well as to increases in final demand $(c_r e^{r})$.
4. A direct effect of the recycling activity to produce the final demand given a new (c_r) on the employment.

The last third of equation (3) tells us that among all of the terms of equation (3), the terms with ambiguous signs are $v_{\mathrm{II}} a_2^{r}(I - BD)^{-1}B\tilde{y}$ and $v_r c_r f_{rr}$. The other terms are positive (or at least nonnegative). This is natural since we are assuming an efficient recycling industry.

In Case 2 no attention is paid to the abatement of pollutant n until recycling comes into operation so pollutant n is reduced only by $a_{nr}x_r$. To depict this situation, we create a submatrix consisting of the first n - 1 rows and columns of $\tilde{A}$ and denoted by $A^{\#}$ and write $[x_1, \cdots, x_{n-1}]'$, $[c_1, \cdots, c_r, -u_{r+1}, \cdots, -u_{n-1}]'$ and $[a_{n1}, \cdots, a_{n\,n-1}]'$ as $x^{\#}$, $g^{\#}$ and $a_n^{\#}$ respectively. Therefore, the system to be considered is:

$$\begin{cases} (I - A^{\#})x^{\#} = g^{\#} \\ a_n^{\#} x^{\#} \le u_n \end{cases} \tag{4}$$

Since $A^{\#}$ is a matrix of technical coefficients of typical Leontief type, it is justifiable to suppose that $(I - A^{\#})^{-1} = B^{\#}$ exists and is non-negative. If $a_n^{\#} B^{\#} g^{\#} = u_n$, $B^{\#} g^{\#}$ solves system (4). Otherwise, we have to change the exogenous vector so as to make system (4) consistent. In this context, it would be reasonable to reduce at least one final demand for goods, other than recycled goods, because a reduction in c_r entails a decrease in the amount of pollutant n used by industry r. By $\ddot{c}_i$ denote the planned cut in c_i, so that the resultant change in output vector $x^{\#}$ is given by $\ddot{c}_i b_i^{\#}$. By choosing $\ddot{c}_i$ which is the minimum of $\{(a_n^{\#} B^{\#} g^{\#} - u_n)/(a_n^{\#} b_1^{\#}), \ldots, (a_n^{\#} B^{\#} g^{\#} - u_n)/(a_n^{\#} b_{n-1}^{\#})\}$, a satisfactory solution can be found.

We now turn to Case 3. Without loss of generality, we can suppose that good r-1 is of the same quality as good r. If this is so, then it might be possible to consolidate both sectors so that the criterion of consistent aggregation is met.[2] Unfortunately such an approach proves impossible irrespective of (A) or (NA), since in this case, the sign of a_{nr-1} differs from that of a_{nr}. This violates the necessary and sufficient conditions for consistent aggregation which require the existence of a scalar k such that $a_{ir-1} = k a_{ir}$ for all i different from both r - 1 and r. Therefore we must content ourselves with letting the first order aggregation bias vanish. If we do so, then Theorem 11-2 of Morimoto (1980; pp. 85 - 87) assures us that our aim can be attained by adopting the aggregation weights defined by[3]

$$w_j = \frac{c_j}{(c_{r-1} + c_r)} \quad ; \qquad j = r-1, r^3$$

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[2] For the notion of consistent aggregation and necessary and sufficient conditions for consistent aggregation, refer to McManus (1956; p.32) and/or Kimura (1985; p.169).

[3] Theorem 11-2 (1980; pp. 186 -187) is, as Morimoto points out, a concise restatement of Theil's assertion (1954; p.119) if we partition the set of all indices ($\{1, \ldots, n\}$) into $P_1, \cdots, P_m$ and suppose that the sectors belonging to P_i are consolidated to be the hybrid sector i(i = 1,...,m). We now introduce additional notation: $\#P_i$ = the number of the elements in P_i, f = a vector of exogenous variables such as final demand, 1_i = a row vector consisting of $\#P_i$ ones, $A_{P_i P_j}$ (f_{P_i}) = a submatrix (subvector) obtained from $A(f)$ by extracting $a_{ht}(f_h)$ such that $h \in P_i$ and $t \in P_j$ and A^* = the matrix of aggregated technical coefficients. In these symbols, the first order aggregation bias is represented as $(A^*S - SA)f$, where S stands for an $m \times n$ matrix satisfying:

$$s_{ij} = \begin{cases} 0 & j \notin P_i \\ 1 & \text{otherwise} \end{cases} \qquad i = 1, \ldots, m$$

Since $((A^*S - SA)f_i) = (a_{i1}^* 1_1 f_{P_1} - 1_i A_{P_i P_1}, \cdots, a_{im} 1_m f_{P_m} - 1_m A_{P_i P_m} f_{P_m})$, the first order aggregation bias is seen to disappear if

$$a_{ij}^* = (1_i A_{P_i P_j} f_{P_j}) / (1_j f_{P_j}) \qquad \text{for } i = 1, \cdots, m \tag{§}$$

holds true. Put $m = n$-1, $P_i = \{i\}$ or $\{i+1\}$ according to whether $i < r$ or $i > r$, and $P_{r-1} = \{r\text{-}1, r\}$ in (§). Then the weights shown in the text follow at once.

These weights are dependent upon the current bill of goods, which implies that in order to keep the first order aggregation bias null the aggregated technical coefficients and hence the aggregated Leontief inverse should be modified as the final demand for goods r - 1 and/or r varies. At a glance, these modifications look tedious. However, taking the constancy of the technical coefficients before the aggregation for granted, the application of the extended Sherman-Morrison formula by Sonis and Hewings (1995, equation (12) on p.64 and its column version) makes this approach practical since the weights are extremely easy to calculate.

It is now evident that the remaining case, Case 4, needs no further consideration, because it is solved in the same manner as was employed in Case 2.

4. CONCLUDING REMARKS

Summarising our analysis, we may conclude that an efficient recycling activity, once it starts, has favourable environmental and demographic effects, although study of the latter is fairly limited at present. Therefore, we urgently need to further our analysis of more complicated input-output systems. In order to do this, we have to rely not only on formal analysis but also on a simulation experiment based on numerical computation. As a prerequisite to this, we probably need to revise the methods we use to automatically generate computer programs to solve a given non-linear dynamic system (Kimura (1993, pp. 300 - 322)) in order that some of the anticipated complexities of the numerical simulation can be handled as automatically as possible.

In principle, the aggregation scheme to cope with situation (S) hardly bothers us. However, if we wish to extend the model to include many recycling activities, producing goods whose quality is close to that of goods already produced then the recalculation concomitant with the aggregation would be cumbersome. In this case the computing method would prove extremely useful, because it automatically generates and runs a program to execute successive applications of the extended Sherman-Morrison formula.

It is also necessary to study the behaviour of real wage-rates as recycling activities have the ability to bring about more employment opportunities. As was pointed out by Negishi (1979; p.93), when households are pessimistic about future increases in labour demand, they are willing to work as much as necessary, provided the prevailing real wage-rate lies within a certain range. As employment opportunities increase the pessimism turns to optimism. Thus, we need to find ways of developing the extended input-output system to cope with patterns of labour supply existing when households are optimistic about employment opportunities

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