ANALYSING A BI-REGIONAL EXTENDED INPUT-OUTPUT MODEL INCORPORATING A LABOUR MARKET ACCOUNT

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ABSTRACT The last thirty years have seen an increasing development and use of extended input-output models, in which the household sector is disaggregated to differentiate between various type of household. The majority of this work has been developed at a single region level, and has recently included the incorporation of labour market accounts to analyse the relationship between labour supply and labour demand. In this paper we extend this type of analysis from a single region model to a bi-regional model. We outline a prototype model and highlight the incorporation of commuting patterns as a relatively new development in this sort of model. The solution of the model is derived, in general form, and some observations made about the implications of the solution. We conclude by analysing the characteristics of some of the variables contained within the inverse, and raising some more general questions about possible limitations of the model.

1. INTRODUCTION

Stemming from the pioneering work of Miernyk *et al.* (1967) the last thirty years have seen an ever increasing development and use of extended input-output models. Extended input-output models refer to a class of input-output model in which the household sector is disaggregated to differentiate between various type of household. Commonly this differentiation distinguishes between employed, unemployed and migrant workers (see, for example, Madden and Batey, 1986; Batey and Rose, 1990; Oosterhaven and Dewhurst, 1990 and Hewings and Madden, 1995). The majority of this work is developed at a single region level, a noticeable exception being the bi-regional model developed by Madden and Trigg (1990).

More recently Batey, Madden and Thomson (1996) have shown that such extended input-output models can be further developed by incorporating labour market accounts (Tyler and Rhodes, 1989) to analyse the relationship between labour supply and labour demand. In this paper we seek to extend the Batey, Madden and Thomson analysis from a single region model to a bi-regional model relying heavily on the specification used by Madden and Trigg (1990).

In the following section we outline a prototype model that combines a bi-regional extended input-output structure with a set of labour market accounts for the two regions. The attraction of conjoining the two research strands is that attention can be focussed on the relationships between the demand for labour, migration patterns, commuting patterns and the size of the labour supply pool. In particular we would highlight the incorporation of commuting patterns to be a relatively new development in this type of model. As a result, if the model were to be implemented in this or an expanded form, it might be most instructively applied in situations where there is both significant migration and significant commuting between the regions.

Given the nature of labour market accounts, which refer to changes in populations and their components over a period of time, the model is best regarded as an attempt to represent the change in a bi-regional demo-economic system in a given period. For ease of exposition a number of simplifying assumptions are made, though, in most cases, we do not consider these to detract from the qualitative import of the model. The model has been constructed from the stand point of analysing a system in which there is growth. It may be that further research will suggest that it is not as attractive a model in times of decline.

In the third section of the paper the solution of the model is derived, in general form, and some observations made about the implications of the solution. We conclude by raising some more general questions about possible limitations of the model and how the model might, at least in theory, be developed and improved.

2. THE MODEL

2.1 Output

We start with the observation that as the model developed in this paper is intended to be a prototype for empirical exercises investigating the changes that take place in regions over a period time, the model expounded in this section is expressed in change form and not in levels.

One version of the standard formulation of the output accounting identities in a bi-regional input-output structure may be written as

$$\begin{aligned}&(I-A_{11})\Delta X_1 - A_{12}\Delta X_2 - h_{1E11}\Delta Y_{E11} - h_{1E22}\Delta Y_{E22} - h_{1C12}\Delta Y_{C12} - h_{1C21}\Delta Y_{C21}\\ &- h_{1U1}\Delta Y_{U1} - h_{1U2}\Delta Y_{U2} - h_{1M12}Y_{M12} - h_{1M21}Y_{M21} - h_{1MX2}Y_{MX2} - h_{1MX1}Y_{MX1} = \Delta F_1 \quad (1)\end{aligned}$$

and

$$\begin{aligned}&- A_{21}\Delta X_1 + (I-A_{22})\Delta X_2 - h_{2E11}\Delta Y_{E11} - h_{2E22}\Delta Y_{E2} - h_{2C12}\Delta Y_{C12} - h_{2C21}\Delta Y_{C21}\\ &- h_{2U1}\Delta Y_{U1} - h_{2U2}\Delta Y_{U2} - h_{2M12}Y_{M12} - h_{2M21}Y_{M21} - h_{2MX2}Y_{MX2} - h_{2MX1}Y_{MX1} = \Delta F_2 \quad (2)\end{aligned}$$

where $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is the bi-regional ($2n$ x $2n$) matrix of direct coefficients (see Appendix 1 for the variable definitions).

Equations 1 and 2 represent the usual Leontief mechanism for a two-region system, with extended consumption entries for locally employed workers in each region, commuters from each region to the other, and unemployed workers in each region. In addition it includes additional migration costs faced by both inter-regional migrants and those from the Rest of the World.

2.2 The Labour Market

The Labour Pool

In the bi-regional model, those in the "labour pool" in Region 1, W_1 may be classified into four groups:

(1) Those employed in the region $i'E_{11}$, where E_{11} is an n-dimensional column vector of residents of Region 1 employed by the industries in Region 1
(2) Those employed in the other region - commuters from Region 1 to Region 2 - $i'C_{12}$, where C_{12} is an n-dimensional column vector of residents of Region 1 employed by the industries in Region 2
(3) Those registered as unemployed and receiving benefit U_1
(4) Potential workers who are not registered as unemployed and who receive no benefit P_1

For ease of exposition we assume that the only commuting is between Regions 1 and 2 and does not take place between the system and the Rest of the World. Clearly

$$W_1 = i'E_{11} + i'C_{12} + U_1 + P_1 \tag{3}$$

Thus over any period

$$\Delta W_1 = i'\Delta E_{11} + i'\Delta C_{12} + \Delta U_1 + \Delta P_1 \tag{4}$$

The change in the labour pool may also be decomposed as

$$\Delta W_1 = B_1 - R_1 + M_{21} - M_{12} + M_{X1} \tag{5}$$

where B_1 is the new entrants to the labour pool, excluding in-migrants, R_1 is the retirals from the labour pool, including out-migrants to the Rest of the World, and M_{jk} is the gross migration flows of persons to the labour pool in k from j to k.

Thus a variant of the labour market account identity is given by

$$i'\Delta E_{11} + i'\Delta C_{12} + \Delta U_1 + \Delta P_1 - M_{21} + M_{12} - M_{X1} = B_1 - R_1 \tag{6}$$

A similar relationship must hold for Region 2

$$i'\Delta E_{22} + i'\Delta C_{21} + \Delta U_2 + \Delta P_2 - M_{12} + M_{21} - M_{X2} = B_2 - R_2 \tag{7}$$

The Creation of Jobs

Jobs become available for two reasons:

(1) Retirements from Employment. For Region 1 these are R_{E1}, an n-dimensional column vector containing retirals by industry, which is assumed exogenous.

(2) Increases due to stimuli to the economies. For Region 1 these are given by

$$\Delta E_1 = \lambda_1 \Delta X_1 \tag{8}$$

where E_1 is an n-dimensional column vector of employment levels by industry in Region 1, and λ_1 is an $(n \times n)$ diagonal matrix of employment-output ratios in the industries in Region 1.

So in any period the number of employment vacancies (an n-dimensional column vector) is given by

$$V_1 = \lambda_1 \Delta X_1 + R_{E1} \tag{9}$$

and for Region 2 by

$$V_2 = \lambda_2 \Delta X_2 + R_{E2} \tag{10}$$

How Jobs are Filled

Suppose there are employment vacancies in Region 1. These will be filled either by:

(1) local persons

(2) persons in Region 2

(3) in-migrants to the system.

We assume that the proportions of the vacancies filled by these categories are α_{11}, α_{21} and $(1 - \alpha_{11} - \alpha_{21})$ respectively.

(1) Local persons may have been

- (1a) unemployed
- (1b) potential workers
- (1c) a new entrant to the labour market

We suppose that the proportions of the vacancies filled by these categories are $\alpha_{11}\beta_{1U}$, $\alpha_{11}\beta_{1P}$ and $\alpha_{11}(1 - \beta_{1U} - \beta_{1P})$.

(2) Similarly persons in Region 2 may have been

- (2a) unemployed
- (2b) potential workers
- (2c) a new entrant to the labour market.

We suppose that the proportions of the vacancies filled by these categories are $\alpha_{21}\gamma_{2U}$, $\alpha_{21}\gamma_{2P}$ and $\alpha_{21}(1 - \gamma_{2U} - \gamma_{2P})$. Note that such people may choose to commute from Region 2 to Region 1 or migrate from Region 2 to Region 1. For ease of exposition we assume that the propensity to commute is the same c_{21} for each group. We further suppose that any migrant from Region 2 to Region 1 takes with them ϕ_{21} additional potential workers.

(3) In-migrants to the system may either choose to locate in Region 1 with probability $(1 - c_{X1})$ or locate in Region 2 and commute with probability c_{X1}. We assume that a migrant into the system will bring with them ϕ_M additional potential workers.

Similar reasoning will apply to the filling of a new job in Region 2.

Non-Employment Related Migration

We allow for the possibility that the unemployed members of the labour force may migrate between Regions 1 and 2, without changing their employment status. We assume that the gross flows depend on the number of vacancies and are given by

$$\eta_2 i' V_1 = \text{unemployed movers from Region 2 to Region 1}$$
$$\eta_1 i' V_2 = \text{unemployed movers from Region 1 to Region 2}$$

New Entrants

New entrants to the labour market in Region 1, that do not find employment may either become unemployed or potential workers. The number that do not find employment is

$$B_1 - \alpha_{11}(1 - \beta_{1U} - \beta_{1P}) i' V_1 - \alpha_{12}(1 - \gamma_{1U} - \gamma_{1P}) i' V_2$$

We assume a proportion μ_1 of these become unemployed and the remaining proportion become potential workers $(1 - \mu_1)$.

Changes

(1) Change in locally employed ΔE_{11}

$$\Delta E_{11} = -R_{E1} + \alpha_{11} V_1 + (1 - c_{21}) \alpha_{21} V_1 + (1 - c_{X1})(1 - \alpha_{11} - \alpha_{21}) V_1 \quad (11)$$

(2) Change in commuting from Region 1 to 2.

$$\Delta C_{12} = c_{12} \alpha_{12} V_2 + c_{X2}(1 - \alpha_{22} - \alpha_{12}) V_2 \quad (12)$$

(3) Change in unemployment

$$\begin{aligned} \Delta U_1 = & -R_{U1} + \mu_1 \{B_1 - \alpha_{11}(1 - \beta_{1U} - \beta_{1P}) i' V_1 - \alpha_{12}(1 - \gamma_{1U} - \gamma_{1P}) i' V_2\} \\ & - \eta_1 i' V_2 + \eta_2 i' V_1 - \alpha_{11} \beta_{1U} i' V_1 - \alpha_{12} \gamma_{1U} i' V_2 \end{aligned} \quad (13)$$

(4) Change in Potential Workers

$$\begin{aligned}\Delta P_1 = & -R_{P1} + (1-\mu_1)\{B_1 - \alpha_{11}(1-\beta_{1U}-\beta_{1P})i'V_1 \\ & -\alpha_{12}(1-\gamma_{1U}-\gamma_{1P})i'V_2\} - \alpha_{11}\beta_{1P}i'V_1 - \alpha_{12}\gamma_{1P}i'V_2 \\ & +\phi_{21}(1-c_{21})\alpha_{21}i'V_1 + \phi_X(1-c_{X1})(1-\alpha_{11}-\alpha_{21})i'V_1 \\ & -\phi_{12}(1-c_{12})\alpha_{12}i'V_2 + \phi_X c_{X2}(1-\alpha_{22}-\alpha_{12})i'V_2\end{aligned} \tag{14}$$

(5) Gross Migration from Region 1 to Region 2

$$M_{12} = \eta_1 i'V_2 + (1+\phi_{12})(1-c_{12})\alpha_{12}i'V_2 \tag{15}$$

(6) Gross Migration from Region 1 to Region 2

$$M_{21} = \eta_2 i'V_1 + (1+\phi_{21})(1-c_{21})\alpha_{21}i'V_1 \tag{16}$$

(7) Migration from The Rest of the World to Region 1

$$M_{X1} = (1-\phi_X)\{(1-c_{X1})(1-\alpha_{11}-\alpha_{21})i'V_1 + c_{X2}(1-\alpha_{22}-\alpha_{12})i'V_2\} \tag{17}$$

It can be shown that these seven relationships do conform to the labour market account identity. Thus the identity may be used in a model instead of one of these equations and in what follows the equation for the change in potential workers is dropped.

For ease of exposition we may reparameterise the remaining equations as follows:

(1) Change in Locally Employed

$$\begin{aligned}\Delta E_{11} &= -R_{E1} + \alpha_{11}V_1 + (1-c_{21})\alpha_{21}V_1 + (1-c_{X1})(1-\alpha_{11}-\alpha_{21})V_1 \\ &= -R_{E1} + V_1 - c_{21}\alpha_{21}V_1 - c_{X1}(1-\alpha_{11}-\alpha_{21})V_1 \\ &= a_1V_1 - R_{E1}\end{aligned} \tag{18}$$

where

$$a_1 = 1 - c_{21}\alpha_{21} - c_{X1}(1-\alpha_{11}-\alpha_{21}) \tag{19}$$

Similarly for Region 2

$$\Delta E_{22} = a_2V_2 - R_{E2} \tag{20}$$

(2) Change in Commuting from Region 1 to 2.

$$\begin{aligned}\Delta C_{12} &= c_{12}\alpha_{12}V_2 + c_{X2}(1-\alpha_{22}-\alpha_{12})V_2 \\ &= (1-a_2)V_2\end{aligned} \tag{21}$$

Similarly for Region 2

$$\Delta C_{21} = (1-a_1)V_1 \tag{22}$$

(3) Change in Unemployment

$$\begin{aligned}\Delta U_1 &= -R_{U1} + \mu_1\{B_1 - \alpha_{11}(1-\beta_{1U}-\beta_{1P})i'V_1 - \alpha_{12}(1-\gamma_{1U}-\gamma_{1P})i'V_2\} \\ &\quad - \eta_1 i'V_2 + \eta_2 i'V_1 - \alpha_{11}\beta_{1U}i'V_1 - \alpha_{12}\gamma_{1U}i'V_2 \\ &= b_{11}i'V_1 + b_{12}i'V_2 + \mu_1 B_1 - R_{U1}\end{aligned} \tag{23}$$

where

$$b_{11} = \eta_2 - \mu_1\alpha_{11}(1-\beta_{1U}-\beta_{1P}) - \alpha_{11}\beta_{1U} \tag{24}$$

$$b_{12} = -\eta_1 - \mu_1\alpha_{12}(1-\gamma_{1U}-\gamma_{1P}) - \alpha_{12}\gamma_{1U} \tag{25}$$

Similarly for Region 2

$$\Delta U_2 = b_{21}i'V_1 + b_{22}i'V_2 + \mu_2 B_2 - R_{U2} \tag{26}$$

(4) Change in Potential Workers

$$\Delta P_1 = -i'\Delta E_{11} - i'\Delta C_{12} - \Delta U_1 + M_{21} - M_{12} + M_{X1} + B_1 - R_1 \tag{27}$$

and for Region 2

$$\Delta P_2 = -i'\Delta E_{22} - i'\Delta C_{21} - \Delta U_2 + M_{12} - M_{21} + M_{X2} + B_2 - R_2 \tag{28}$$

(5) Gross Migration from Region 1 to Region 2

$$\begin{aligned}M_{12} &= \eta_1 i'V_2 + (1+\phi_{12})(1-c_{12})\alpha_{12}i'V_2 \\ &= \{\eta_1 + \phi_{12}(1-c_{12})\alpha_{12}\}\, i'V_2 + (1-c_{12})\alpha_{12}i'V_2\end{aligned} \tag{29}$$

The first of these terms refers to migrants that do not take jobs after migrating and the second to those that do, i.e.

$$M_{12} = M_{12O} + i'M_{12E} \tag{30}$$

where M_{12E} is an n-dimensional column vector of the employment of migrants from Region 1 to Region 2 by the industries in Region 2. We may write

$$M_{12O} = m_{12O}i'V_2 \tag{31}$$

where

$$m_{12O} = \eta_1 + \phi_{12}(1-c_{12})\alpha_{12} \tag{32}$$

and

$$M_{12E} = m_{12E}V_2 \tag{33}$$

where

$$m_{12E} = (1-c_{12})\alpha_{12} \tag{34}$$

(6) Gross Migration from Region 2 to Region 1
Using arguments similar to those immediately above we may write

$$M_{21} = M_{21O} + i'M_{21E} \tag{35}$$

where M_{12E} is an n-dimensional column vector of the employment of migrants from Region 2 to Region 1 by the industries in Region 1.

$$M_{21O} = m_{21O}i' \tag{36}$$

where

$$m_{21O} = \eta_2 + \phi_{21}(1 - c_{21})\alpha_{21} \tag{37}$$

and

$$M_{21E} = m_{21E}V_1 \tag{38}$$

where

$$m_{21E} = (1 - c_{21})\alpha_{21} \tag{39}$$

(7) Migration from the Rest of the World to Region 1

$$M_{X1} = (1 + \phi_X)i'\{(1 - c_{X1})(1 - \alpha_{11} - \alpha_{21})V_1 + c_{X2}(1 - \alpha_{22} - \alpha_{12})V_2\} \tag{40}$$

In what follows it helps to decompose this migration as :

$$M_{X1} = (1 + \phi_X)i'M_{X1E1} + (1 + \phi_X)i'M_{X1E2} \tag{41}$$

where M_{X1E1} is an n-dimensional column vector of the number of in-migrants from the Rest of the World to Region 1 who are subsequently employed in the various industries in Region 1, and M_{X1E2} is an n-dimensional column vector of the number of in-migrants from the Rest of the World to Region 1 who are subsequently employed in the various industries in Region 2.

$$M_{X1E1} = k_{11} \tag{42}$$

$$M_{X1E2} = k_{12}V_2 \tag{43}$$

where

$$k_{11} = (1 + c_{X1})(1 - \alpha_{11} - \alpha_{21}) \tag{44}$$

$$k_{12} = c_{X2}(1 - \alpha_{22} - \alpha_{12}) \tag{45}$$

Similarly for Region 2

$$M_{X2} = (1 + \phi_X)i'M_{X2E1} + (1 + \phi_X)i'M_{X2E2} \tag{46}$$

$$M_{X2E1} = k_{21}V_1 \tag{47}$$

$$M_{X2E2} = k_{22}V_2 \tag{48}$$

where

$$k_{22} = (1 + c_{X2})(1 - \alpha_{22} - \alpha_{12}) \tag{49}$$

$$k_{21} = c_{X1}(1 - \alpha_{11} - \alpha_{21}) \tag{50}$$

2.3 Income

First note that we assume that all employees in any industry in a region obtain the same average earnings regardless of whether they are locally resident or whether they commute. We further assume that the unemployment benefit rate per unemployed worked is the same in the two regions.

The change in income of each of the groups identified in the output equations are determined according to

$$\Delta Y_{E11} - w_1 \Delta E_{11} = \Delta Z_{E11} \tag{51}$$

where w_1 is an n-dimensional row vector of earnings per employee in the industries in Region 1, and Z_{E11} is any non-earned income of the residents of Region 1 employed in Region 1.

$$\Delta Y_{E22} - w_2 \Delta E_{22} = \Delta Z_{E22} \tag{52}$$

where w_2 is an n-dimensional row vector of earnings per employee in the industries in Region 2, and Z_{E22} is any non-earned income of the residents of Region 2 employed in Region 2.

$$\Delta Y_{C12} - w_2 \Delta C_{12} = \Delta Z_{C12} \tag{53}$$

where Z_{C12} is any non-earned income of the residents of Region 1 employed in Region 2.

$$\Delta Y_{C21} - w_1 \Delta C_{21} = \Delta Z_{C21} \tag{54}$$

where Z_{C21} is any non-earned income of the residents of Region 2 employed in Region 1.

$$\Delta Y_{U1} - s \Delta U_1 = \Delta Z_{U1} \tag{55}$$

where s is the unemployment benefit rate and Z_{U1} is any non-benefit income of unemployed persons in Region 1.

$$\Delta Y_{U2} - s \Delta U_2 = \Delta Z_{U2} \tag{56}$$

where Z_{U2} is any non-benefit income of unemployed persons in Region 2.

$$Y_{M12} - w_2 M_{12E} - s f_{12} M_{12O} = Z_{M12} \tag{57}$$

where

$$f_{12} = \frac{\eta_1}{\eta_1 + \phi_{12}(1 - c_{12})\alpha_{12}} \tag{58}$$

and Z_{M12} is any non earned or non-benefit income of the migrants from Region 1 to Region 2.

$$Y_{M21} - w_1 M_{21E} - s f_{21} M_{21O} = Z_{M21} \tag{59}$$

where

$$f_{21} = \frac{\eta_2}{\eta_2 + \phi_{21}(1 - c_{21})\alpha_{21}} \tag{60}$$

and Z_{M21} is any non earned or non-benefit income of the migrants from Region 2 to Region 1.

$$Y_{MX1} - w_1 M_{X1E1} - w_2 M_{X1E2} = Z_{MX1} \tag{61}$$

where Z_{MX1} is any non earned income of the migrants from the Rest of the World to Region 1.

$$Y_{MX2} - w_1 M_{X2E1} - w_2 M_{X2E2} = Z_{MX2} \tag{62}$$

where Z_{MX2} is any non earned income of the migrants from the Rest of the World to Region 2.

3. THE SOLUTION OF THE MODEL

The equation system that is described in the previous section can be represented by a matrix equation of the form:

$$\boldsymbol{C}\,\boldsymbol{Y} = \boldsymbol{Z} \tag{63}$$

where $\boldsymbol{Y}$ is the column vector of the endogenous variables, i.e.

$$\begin{aligned}\boldsymbol{Y} = [&\Delta \boldsymbol{X}_1, \Delta \boldsymbol{X}_2, \Delta Y_{E11}, \Delta Y_{E22}, \Delta Y_{C12}, \Delta Y_{C21}, \Delta Y_{U1}, \Delta Y_{U2}, \Delta Y_{M12}, \Delta Y_{M21},\\ &\Delta Y_{MX1}, \Delta Y_{MX2}, \Delta \boldsymbol{E}_{11}, \Delta \boldsymbol{E}_{22}, \Delta \boldsymbol{C}_{12}, \Delta \boldsymbol{C}_{21}, \Delta U_1, \Delta U_2, \boldsymbol{M}_{12E}, \boldsymbol{M}_{12O},\\ &\boldsymbol{M}_{12}, \boldsymbol{M}_{21E}, \boldsymbol{M}_{21O}, \boldsymbol{M}_{21}, \boldsymbol{M}_{X1E1}, \boldsymbol{M}_{X1E2}, \boldsymbol{M}_{X1}, \boldsymbol{M}_{X2E1}, \boldsymbol{M}_{X2E2}, \boldsymbol{M}_{X2},\\ &\Delta P_1, \Delta P_2, V_1, V_2]'\end{aligned} \tag{64}$$

where the (n-dimensional) vectors are highlighted, and $\boldsymbol{Z}$ is the column vector of exogenous variables, i.e.

$$\begin{aligned}\boldsymbol{Z} = [&\Delta F_1, \Delta F_2, \Delta Z_{E11}, \Delta Z_{E22}, \Delta Z_{C12}, \Delta Z_{C21}, \Delta Z_{U1}, \Delta Z_{U2}, \Delta Z_{M12}, \Delta Z_{M21},\\ &Z_{MX1}, Z_{MX2}, -R_{E1}, -R_{E2}, 0, 0, \mu_1 B_1 - R_{U1}, \mu_2 B_2 - R_{U2}, 0, 0, 0, 0, 0,\\ &0, 0, 0, 0, 0, 0, 0, B_1 - R_1, B_2 - R_2, R_{E1}, R_{E2}]'\end{aligned} \tag{65}$$

For ease of presentation we may partition the matrix of coefficients $\boldsymbol{C}$ as:

$$\begin{bmatrix} (I-A) & -h & 0 \\ 0 & I & -W \\ -L & 0 & K \end{bmatrix}$$

where the actual coefficients in each of the partitions is given in Appendix 2.

However in order to investigate, at least in general terms, the solution of the model we use a 6x6 partition of the form

$$\begin{bmatrix} I-A & -h & 0 & 0 & 0 & 0 \\ 0 & I & -w_1 & -s & -w_2 & 0 \\ 0 & 0 & I & 0 & 0 & -a \\ 0 & 0 & 0 & I & 0 & -b \\ 0 & 0 & 0 & 0 & I-\phi & -m \\ -\lambda & 0 & Y_1 & Y_s & M & I \end{bmatrix}$$

The implicit partitioning of the 3x3 partition to obtain the 6x6 partition is shown by the dotted partition lines in Appendix 2. The general solution of the model specified in this way is shown in the inverse in Appendix 3. Here we see firstly two important variables, R and Q. R aggregates together the income of the various demographic categories. Specifically, starting from the right hand end of the expression, we see that m, the element of the matrix which draws migrants into the system in response to vacancies, is pre-multiplied by $(I - \phi)^{-1}$, which further allocates exogenous in-migrants into one or other of the regions. Pre-multiplication of this by w_2 generates the income of these migrants. The second term, sb, carries out a similar exercise for unemployed workers, while the first term does the same for employed workers and commuters. R therefore represents the total income of the economically active elements of the demographic system.

Q is a more complex variable. The last three elements of the expression have the effect of aggregating the employed, unemployed, commuters and migrants into their final regions of location, while the second term generates job demand from total demographic consumption. Subtracting change in workers, unemployed and migrants from change in labour demand provides a first round balance of vacancies and change in potential workers: Q^{-1} therefore represents the total number of vacancies generated in the system as a result of demographic consumption after changes in the economically active population have been taken into account.

Q^{-1} is clearly a pivotal variable in the model (see Appendix 3). The sixth column, representing the effect upon different variables of the system of changes in retirals and "births", consists of fairly straightforward variations on the theme of Q^{-1}. For example, the fifth term in this column generates migrants from Q^{-1} via m, then allocates them to their destination regions via $(I - \phi)^{-1}$. The fourth term does the same for unemployed workers, and the third term the same for employed workers. Term two generates the total income of all workers generated from the retiral and birth changes, and term one runs that income through the consumption vector h and the Leontief industrial part of the model to derive changes in gross outputs.

Other columns may be interpreted in similar ways. The first column, for example, transforms industrial final demands through $\lambda(I - A)^{-1}$ into labour demand, which is then transformed by Q^{-1} into vacancies, and hence, depending on the premultiplying terms, gross outputs, incomes, employed workers, commuters and migrants. Column two represents the effects upon the system of changes in exogenous income, reflected in Appendix 3 by the post-multiplication of the terms by the vector h. Exogenous injections of employed, unemployed and migrant workers have an attenuating effect, although interpretation here, as always, is in economic/demographic terms obscure.

Further research will derive an empirical version of the model, and will include as well as scenario testing and multiplier analysis the inspection and analysis of the terms Q and R, and other elements of the inverse. Possible further extensions of the model would involve endogenisation of the demographic side of the system by including a cohort survival model within the matrix structure. This would enable explicit modelling of entry to and exit from the labour market due to ageing, and would permit relationships between migration and population to be modelled endogenously.

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APPENDIX 1: Definitions of Variables

X_r	n-dimensional column vector of the gross outputs of industries in Region r (r = 1, 2)
h_{1E11}	n-dimensional column vector of the proportion of their income that persons resident in Region 1 and employed in Region 1 spend on the outputs of the industries of Region 1
h_{2E11}	n-dimensional column vector of the proportion of their income that persons resident in Region 1 and employed in Region 1 spend on the outputs of the industries of Region 2
Y_{E11}	total income of persons resident in Region 1 and employed in Region 1
h_{1E22}	n-dimensional column vector of the proportion of their income that persons resident in Region 2 and employed in Region 2 spend on the outputs of the industries of Region 1
h_{2E22}	n-dimensional column vector of the proportion of their income that persons resident in Region 2 and employed in Region 2 spend on the outputs of the industries of Region 2
Y_{E22}	total income of persons resident in Region 2 and employed in Region 2
h_{1C12}	n-dimensional column vector of the proportion of their income that persons resident in Region 1 and employed in Region 2 spend on the outputs of the industries of Region 1
h_{2C12}	n-dimensional column vector of the proportion of their income that persons resident in Region 1 and employed in Region 2 spend on the outputs of the industries of Region 2
Y_{C12}	total income of persons resident in Region 1 and employed in Region 2 (i.e. commuters from Region 1 to Region 2)
h_{1C21}	n-dimensional column vector of the proportion of their income that persons resident in Region 2 and employed in Region 1 spend on the outputs of the industries of Region 1
h_{2C21}	n-dimensional column vector of the proportion of their income that persons resident in Region 2 and employed in Region 1 spend on the outputs of the industries of Region 2
Y_{C21}	total income of persons resident in Region 2 and employed in Region 1 (i.e. commuters from Region 2 to Region 1)
h_{1U1}	n-dimensional column vector of the proportion of their income that unemployed persons resident in Region 1 spend on the outputs of the industries of Region 1

h_{2U1} n-dimensional column vector of the proportion of their income that unemployed persons in Region 1 spend on the outputs of the industries of Region 2

Y_{U1} total income of unemployed persons in Region 1

h_{1U2} n-dimensional column vector of the proportion of their income that unemployed persons resident in Region 2 spend on the outputs of the industries of Region 1

h_{2U2} n-dimensional column vector of the proportion of their income that unemployed persons in Region 2 spend on the outputs of the industries of Region 2

Y_{U2} total income of unemployed persons in Region 2

h_{1M12} n-dimensional column vector of the additional proportion of their income that migrants from Region 1 to Region 2 spend on the outputs of the industries of Region 1

h_{2M12} n-dimensional column vector of the additional proportion of their income that migrants from Region 1 to Region 2 spend on the outputs of the industries of Region 2

Y_{M12} total income of migrants from Region 1 to Region 2

h_{1M21} n-dimensional column vector of the additional proportion of their income that migrants from Region 2 to Region 1 spend on the outputs of the industries of Region 1

h_{2M21} n-dimensional column vector of the additional proportion of their income that migrants from Region 2 to Region 1 spend on the outputs of the industries of Region 2

Y_{M21} total income of migrants from Region 2 to Region 1

h_{1MX1} n-dimensional column vector of the additional proportion of their income that migrants from the Rest of the World to Region 1 spend on the outputs of the industries of Region 1

h_{2MX1} n-dimensional column vector of the additional proportion of their income that migrants from the Rest of the World to Region 1 spend on the outputs of the industries of Region 2

Y_{MX1} total income of migrants from the Rest of the World to Region 1

h_{1MX2} n-dimensional column vector of the additional proportion of their income that migrants from the Rest of the World to Region 2 spend on the outputs of the industries of Region 1

h_{2MX2} n-dimensional column vector of the additional proportion of their income that migrants from the Rest of the World to Region 2 spend on the outputs of the industries of Region 2

Y_{MX2} total income of migrants from the Rest of the World to Region 2

F_1 n-dimensional column vector of final demands for the outputs of the industries in Region 1

F_2 n-dimensional column vector of final demands for the outputs of the industries in Region 2.

APPENDIX 2: Elements of the 3x3 Partitioned Matrix of Coefficients

$$(I-A)=\begin{bmatrix} I-A_{11} & -A_{12} \\ -A_{21} & I-A_{22} \end{bmatrix}$$

$$h=\begin{bmatrix} h_{1E11} & h_{1E22} & h_{1C12} & h_{1C21} & h_{1U1} & h_{1U2} & h_{1M12} & h_{1M21} & h_{1MX1} & h_{1MX2} \\ h_{2E11} & h_{2E22} & h_{2C12} & h_{2C21} & h_{2U1} & h_{2U2} & h_{1M12} & h_{2M21} & h_{2MX1} & h_{2MX2} \end{bmatrix}$$

$$W=\left[\begin{array}{cccc|cc|cccccccccccc|cccc}
w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & w_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & w_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & w_2 & s.f_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_1 & s.f_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_1 & w_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_1 & w_2 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

$$L'=\left[\begin{array}{cccc|cc|ccccccccccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2
\end{array}\right]$$

$$
K = \left[\begin{array}{cccc|cc|cccccccccccc|cccc}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_1.I & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_2.I \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (a_2-1).I \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (a_1-1).I & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_{11}.i' & -b_{12}.i' \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_{21}.i' & -b_{22}.i' \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -m_{12E}.I \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -m_{12O}.i' \\
0 & 0 & 0 & 0 & 0 & 0 & -i' & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -m_{21E}.I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -m_{21O}.i' & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i' & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_{11}.I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_{12}.I \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (-1-\phi_X).i' & (-1-\phi_X).i' & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & -k_{21}.I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & -k_{22}.I \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (-1-\phi_X).i' & (-1-\phi_X).i' & 1 & 0 & 0 & 0 & 0 \\
\hline
i' & 0 & i' & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & i' & 0 & i' & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I
\end{array}\right]
$$

APPENDIX 3: The General Solution of the 6x6 Partitioned Model

$$
\left[\begin{array}{cccc}
(I-A)^{-1}[I+h.R.Q^{-1}.\lambda(I-A)^{-1}] & (I-A)^{-1}h(I+R)Q^{-1}.\lambda(I-A)^{-1}h & (I-A)^{-1}h[w_1+R.Q^{-1}\{\lambda(I-A)^{-1}h.w_1-Y_I\}] & \ldots \\
R.Q^{-1}.\lambda(I-A)^{-1} & (I+R)Q^{-1}.\lambda(I-A)^{-1}h & w_1+R.Q^{-1}\{\lambda(I-A)^{-1}h.w_1-Y_I\} & \ldots \\
a.Q^{-1}.\lambda(I-A)^{-1} & a.Q^{-1}.\lambda(I-A)^{-1}h & I+a.Q^{-1}\{\lambda(I-A)^{-1}h.w_1-Y_I\} & \ldots \\
b.Q^{-1}.\lambda(I-A)^{-1} & b.Q^{-1}.\lambda(I-A)^{-1}h & b.Q^{-1}\{\lambda(I-A)^{-1}h.w_1-Y_I\} & \ldots \\
(I-\phi)^{-1}m.Q^{-1}.\lambda(I-A)^{-1} & (I-\phi)^{-1}m.Q^{-1}.\lambda(I-A)^{-1}h & (I-\phi)^{-1}m.Q^{-1}\{\lambda(I-A)^{-1}h.w_1-Y_I\} & \ldots \\
Q^{-1}.\lambda(I-A)^{-1} & Q^{-1}.\lambda(I-A)^{-1}h & Q^{-1}\{\lambda(I-A)^{-1}h.w_1-Y_I\} & \ldots
\end{array}\right.
$$

$$
\left.\begin{array}{ccc}
(I-A)^{-1}h[s+R.Q^{-1}\{\lambda(I-A)^{-1}h.s-Y_s\}] & (I-A)^{-1}h[w_2(I-\phi)^{-1}+R[Q^{-1}.\lambda(I-A)^{-1}h.w_2(I-\phi)^{-1}-M(I-\phi)^{-1}] & (I-A)^{-1}h.R.Q^{-1} \\
s+R.Q^{-1}\{\lambda(I-A)^{-1}h.s-Y_s\} & w_2(I-\phi)^{-1}+R[Q^{-1}.\lambda(I-A)^{-1}h.w_2(I-\phi)^{-1}-M(I-\phi)^{-1}] & R.Q^{-1} \\
a.Q^{-1}\{\lambda(I-A)^{-1}h.s-Y_s\} & a[Q^{-1}.\lambda(I-A)^{-1}h.w_2(I-\phi)^{-1}-M(I-\phi)^{-1}] & a.Q^{-1} \\
I+b.Q^{-1}\{\lambda(I-A)^{-1}h.s-Y_s\} & b[Q^{-1}.\lambda(I-A)^{-1}h.w_2(I-\phi)^{-1}-M(I-\phi)^{-1}] & b.Q^{-1} \\
(I-\phi)^{-1}m.Q^{-1}\{\lambda(I-A)^{-1}h.s-Y_s\} & (I-\phi)^{-1}[I+m.Q^{-1}.\lambda(I-A)^{-1}h.w_2(I-\phi)^{-1}-M(I-\phi)^{-1}] & (I-\phi)^{-1}.m.Q^{-1} \\
Q^{-1}\{\lambda(I-A)^{-1}h.s-Y_s\} & Q^{-1}.\lambda(I-A)^{-1}h.w_2(I-\phi)^{-1}-M(I-\phi)^{-1} & Q^{-1}
\end{array}\right]
$$

where

$$Q=[I-\lambda(I-A)^{-1}h.R+Y_I.a+Y_s.b+M(I-\phi)^{-1}m]$$

and

$$R=[w_1.a+s.b+w_2(I-\phi)^{-1}m]$$