TOWARD A GENERALISED DUAL APPROACH TO VON THÜNEN TYPE LOCATION MODELS

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ABSTRACT This paper applies duality theory to a generalised Von Thünen model of location. Generalised Von Thünen models are defined as those location models where there is economic attraction in location to some central place but competition for central location leads to decentralising forces in location through the price of goods fixed in location. The paper extends duality analysis within the location paradigm to new dual problems of the direct choice of location to minimise or maximise location relative to central place. The paper finds that the optimal value functions associated with these problems are potentially useful avenues for comparative static analysis in location. Further, the dual approach enables the identification of alternative conditional demand functions for location fixed goods which leads to a potentially useful decomposition in the examination of the influence of location on the demand for such goods.

1. INTRODUCTION

Samuelson (1983, pp.1468-1469), rightly indicates that Von Thünen's "Isolated State" anticipated marginalism and general equilibrium. Moreover, Samuelson shows unfortunately that these contributions have not been fully recognised in mainstream economics. In urban economics the story is different. Von Thünen's contribution to location theory is well recognised and his agricultural model with its familiar rings or concentric zones of cultivation is seen as an important antecedent to the development of intra-urban models of location equilibrium, Alonso (1960 and 1964). By the mid 1970s the rapid development of urban residential models had led to a readily accepted generalisation of residential location theory based on a Von Thünen type monocentric city. For the elements of such a model see Richardson (1977, pp.254-258). More recent developments of the urban residential model have been couched in terms of duality in optimising issues associated with residential location. Perhaps the most comprehensive application of duality in intra-urban residential location, following an Alonso type model, is Fujita (1989, chapter 2). Not only does he recast the standard Alonso type residential model in duality utilising the indirect utility and expenditure functions, but he also produces an important dual in the issue, that of maximising offer price (bid rent) conditioned on utility.

The purpose of this paper is to recast the general Von Thünen location model in terms of modern duality theory. It aims to mirror the general approach to duality in modern economic theory taken by Diewert (1982), but apply this approach in a spatial context. The paper is not merely a synthesis of duality theory and urban modelling. Novelty is introduced in the approach through:

- the fact that this is perceived as a general approach to all Von Thünen type location models;
- the derivation of the bid rent function from the optimal value function of the primal problem. This is different from the usual route taken in the literature;
- the utilisation of alternative dual problems such as choosing a location to minimise the difference between bid rent and market rent, or choosing a location directly to minimise or maximise radial distance, all subject to the constraint that bid rent must be at least equal to market rent.
- the derivation of a conditional demand function for a location good such as land.

The aim is to produce a general approach to Von Thünen location models; therefore objective functions, definitions of choice variables and parameters have been left suitably general to include production and consumption problems. A generalised Von Thünen model, as a location model, will have the following characteristics:

Location is relative to some central point, with location forces being equal in every direction, so that location refers to radial distance from that central point.

1. For the locators there is some economic attraction in proximity to the central place (centripetal force).
2. Location relative to the central place requires purchase of a location fixed good, say land.
3. In competition for central location the price of the location fixed good, as a decreasing function of radial distance, acts to disperse locators around the central place (centrifugal force).

2. THE GENERALISED MODEL AND THE APPLICATION OF STANDARD DUALITY THEORY IN LOCATION.

The typical Von Thünen problem can be characterised in the following way:

$$\max_r \left\{ F(\mathbf{a}(r), \mathbf{b}, w) \equiv \max_{\mathbf{x}} \{ f(\mathbf{x}, \mathbf{a}(r), \mathbf{b}) : g(\mathbf{x}, \mathbf{a}(r), \mathbf{b}, w) \leq 0 \} \right\}; \tag{1}$$

or by a dual to (1) as:

$$\min_r \left\{ G(\mathbf{a}(r), \mathbf{b}, w) \equiv \min_{\mathbf{x}} \{ g(\mathbf{x}, \mathbf{a}(r), \mathbf{b}, w) : f(\mathbf{x}, \mathbf{a}(r), \mathbf{b}) \geq 0 \} \right\}; \tag{2}$$

where:

x is a vector of choice variables one of which will be fixed in location.

r is radial distance.

a is a vector of parameters, each of which is a function of distance in such a way that they are a centripetal force. That is, they make central locations attractive.

b is a vector of non-location parameters.

w is the price of the location good and is an unknown function of distance. The price of this good has been isolated from the other parameters in that it will act as a centrifugal force in competition for location. That is it will act as a dispersive location force and will counteract the other location parameters.

The functions f and g could refer to production and cost if the issue is a producer problem or utility and expenditure if the issue is a consumer problem.

In both cases it must be made clear that the distinction between the inner and outer problem is a heuristic device to isolate the location issue from other aspects of a production or consumption problem. The separation into the inner and outer problem is attributable to Anas (1982). It does not indicate that the relevant functions are separable as in say two stage budgeting and a separable utility function, see Gorman (1959) and (1987) and Blackorby *et al* (1978, chapter 3). In this sense space is not to be treated in the same way as time in intertemporal allocation, where say in the consumption model consumers allocate expenditure at successive points in time. Each locator in the Von Thünen model is regarded as only taking up one location in radial distance from the central focal point. Nevertheless, it is feasible to separate the problem as in (1) or (2) because at each location the marginal rate of technical substitution (producer problem) or the marginal rate of substitution (consumer problem) of the choice variables will be independent of location.

However, adapting the nomenclature of intertemporal consumer theory, see McLaren and Cooper (1987), the functions F and G, which are the optimal value functions for the dual inner problems, will be referred to as aspatial, because the problem of location has not been solved in the inner optimisation. When r is optimised out of F and G the resulting optimal value functions will be referred to as spatial functions F_r and G_r. All optimal value functions where r is to be determined, that is are still functions of r, are aspatial functions in the sense that they evaluate an optimisation in which location has not been optimally chosen. They are aspatial in the sense that spatial considerations have not been involved in their optimisation.

3. THE BID RENT (OFFER PRICE) FOR THE LOCATION FIXED GOOD.

Solutions to (1) or (2) are trivial at the moment and will lead to central location until some structure is found for w in r. Rather than impose this structure *a priori*, Von Thünen type models attempt to establish it through competitive bidding within an equilibrium model. Essential to this is the formation of bid rent or offer price as a function of radial distance for locators. A route to bid rent as a function of distance r is through the optimal value functions of (1) and (2).

Firstly, establish that the optimal value functions for the alternative inner problems, F and G, are monotone functions of w and then invert them for w. For F denote the inverse as:

$$\omega(\mathbf{a}(r), \mathbf{b}, f)\ ; \tag{3}$$

and for G,

$$\omega^{G}(\mathbf{a}(r), \mathbf{b}, g)\ . \tag{4}$$

The result (3) or (4) is a maximum value function, known as the bid rent function, conditioned on levels in f or in g and is the maximum at any given location r a locator can bid for units of the location good. This route to the bid rent curve should be compared to that of Wheaton (1979, p.110), Sasaki (1987, p.55) and Fujita (1989, p.14), in the Alonso type urban residential model. Using the notation here, these works invert the function g for w in (1) and then pose the problem:

$$\omega(\mathbf{a}(r), \mathbf{b}, f) \equiv \max_{\mathbf{x}}\{w(\mathbf{x}, \mathbf{a}(r), \mathbf{b}) : f(\mathbf{x}, \mathbf{a}(r), \mathbf{b}) = f\}\ .$$

Returning to the method adopted here, it is useful to form the identities (5) or (6) below, through substitution of (1) and (2) into (3) and (4) respectively. It is then possible to employ envelope theorems to establish important first and second order properties of the bid rent function from these identities:

$$\omega(\mathbf{a}(r), \mathbf{b}, F(\mathbf{a}(r), \mathbf{b}, w)) \equiv w\ \text{; and} \tag{5}$$

$$\omega^{G}(\mathbf{a}(r), \mathbf{b}, G(\mathbf{a}(r), \mathbf{b}, w)) \equiv w\ . \tag{6}$$

The identities in (5) and (6) make economic sense. If a firm or household has a set of bid rents conditioned on f or g and takes up a location, so that it transacts over the location good, then by the definition of bid rent, the market price it pays must be an element in its bid rent set.

Differentiating (5) with respect to w gives:

$$\frac{\partial}{\partial f}\omega(\mathbf{a}(r), \mathbf{b}, f) = \frac{1}{\partial F / \partial w}\ . \tag{7}$$

Now, using the envelope properties of the function F, say through Hotelling's lemma or Roy's identity, identifies the denominator on the right hand side (or some appropriate variation of it) as the demand for the location good (albeit conditioned on w). For illustration here, it can be supposed that $\partial F / \partial w$ gives a relevant demand expression, say $x^{w}(\mathbf{a}(r),\mathbf{b},w)$.

Now, however, inspection of (7) shows that, in this case, demand can be retrieved as:

$$x^{f}(\mathbf{a}(r),\mathbf{b},f)=\frac{1}{\partial\,\omega(\mathbf{a}(r),\mathbf{b},f)/\partial f}\,. \tag{8}$$

The importance of (8) is that it asserts that a relevant demand, such as the demand for land, retrieved in this way will be a function of the arguments of ω and therefore will be conditioned on f and abstracted from own-price. This opens up the prospect of signing the derivatives of the demand for the location good with respect to its arguments (conditioned on f) even if, in the conventional own-price conditional functions, there may be ambiguities. Whether this prospect can be realised will depend upon the structure of the optimal value function from which ω is obtained by inversion. It might seem unusual that the demand for land derived from the bid rent function is conditioned on levels of the functions F or G and abstracted from own-price. Moreover, because they are highly conditioned, such demand functions may be of limited empirical use. Nevertheless, they will serve a useful function in that they can be used to analyse the relationship between the demand for the location good and radial distance r. Particularly, abstracting from own-price effects on demand allows concentration on the effects of parameters $\mathbf{a}$ on the demand in radial distance. Additionally, and most importantly from an empirical point of view, tests of any demand hypothesis in radial distance may need to be conducted under conditions in which the "own-price" is not available. These alternative conditional functions hold out attractive possibilities for different empirical specifications.

Continuing, the remainder of the first derivative properties of ω in its other arguments can now be examined by straightforward differentiation and substitution. That is for example, take any parameter a, which is an element of $\mathbf{a}$, and any parameter b, an element of $\mathbf{b}$, and differentiate ω with respect to these parameters.

Properties 1 (further first derivative properties of ω)

$$\frac{\partial\,\omega}{\partial\,a}=-\frac{\partial F/\partial a}{\partial F/\partial w}\text{ ; and}$$

$$\frac{\partial\,\omega}{\partial\,b}=-\frac{\partial F/\partial b}{\partial F/\partial w}\,.$$

This route should be compared to say Sasaki (1987, p.55). He uses the first order conditions to the problem of choosing $\mathbf{x}$ to maximise ω, subject to a level in f, to establish the first order properties of ω.

A similar procedure, but using the envelope properties of G, through say Shephard's lemma, establishes the envelope properties of ω^G. The demand for the location good can be retrieved through the envelope properties of ω^G. Again, this will be conditioned on levels in g and abstracted from own-price.

With respect to the second order properties of ω, the discussion following (8) shows clearly that the signs of the first and second derivatives of ω or ω^G are determined by the characteristics of the functions F or G. The first derivative signs are straightforward. The second derivative signs depend on the curvature characteristics of the function F or G, because the functions ω and ω^G are expressions for the level curves of F and G. The function ω is convex if F is quasi-convex in $\mathbf{a}$ and $\mathbf{b}$ and ω is the expression for the lower contour function, that is ω is decreasing in levels of f. The function ω^G is convex if G is quasi-concave in $\mathbf{a}$ and $\mathbf{b}$ and ω^G is the expression for the upper contour function, that is ω^G is increasing in the levels of g (Diewert, 1982 and Cornes, 1992).

Next, it is necessary to establish the curvature properties of ω or ω^G in r. With minimum qualification this can be done when the functions $\mathbf{a}(r)$ play an exclusively centripetal role. In this case, using the fact that ω or ω^G are expressions for the lower or upper contour functions of F or G, so that determining the curvature characteristics of F or G and that F or G are decreasing or increasing functions of w and $\mathbf{a}(r)$, establishes the curvature properties of ω or ω^G in r. Alternatively, one can use the first and second derivative properties of ω and ω^G to establish their curvature characteristics, but again these are ultimately determined by the first and second derivative properties of F and G. When $\mathbf{a}(r)$ plays an exclusively centripetal role ω and ω^G are monotone decreasing convex functions of r.

4. THE STRUCTURE OF MARKET RENT FOR THE LOCATION GOOD IN DISTANCE.

Now one may use (3) or (4) in competitive bidding to establish that market price, $w(r)$ is a monotone decreasing continuous convex function. This is determined by the fact that w is supported from below (competitive bidding) by a family of convex functions ω or ω^G. It must be made clear however, that when w is supported from below by a family of bid rent functions differentiated by their degree of slope in r, the convexity of those bid rent functions is sufficient, but not necessary, for the convexity of w. This observation needs to be elaborated upon in order to allow for the empirically plausible case where not all the parameters $\mathbf{a}(r)$ play a centripetal role or, alternatively, where the conditioning variables of the ω function are, unlike f and g, not able to be directly linked in such a way as to infer convexity (*resp.* quasi-concavity) of ω in $\mathbf{a}$ if ω is decreasing (*resp.* increasing) in the conditioning variable. To handle the more general situation of potential non-convexity (perhaps even concavity) of ω in r, it is helpful to introduce the general notion of F convexity. A careful examination of the definition of generalised F convexity[1] (see Avriel *et al*, (1988), pp.294-295) will reveal that convexity of the support functions is not a necessary condition for the decreasing F convexity of the function $w(r)$.

<u>*Definition 1*</u> *(after Avriel, et al, op. cit.)*

$w(r)$ *is a real function on the set S where* $S \subset \Re_+$.

w *is* F *convex on the set S if for every* $r^0 \in S$ *there is a function* $\omega(r)$ *such that*:

$$w(r^0) = \omega(r^0) \quad and \quad w(r) \geq \omega(r) \quad \forall r \in S \ .$$

Thus, a land or built space rent function which is a monotone decreasing F convex function of radial distance could even be derived from a family of concave bid rent curves as long as:

1. the bid rent curves are monotone decreasing in r; and
2. there is complete non-coincidence of the slopes of the family of support functions, so that $w(r^0) = \omega(r^0)$ and $w(r) > \omega(r)$, $\forall r \in S$, $r_0 \neq r$.

Whether or not an F convex market rent function defined in this way is in fact convex in the standard sense does not depend on the curvature properties of the support function (although convexity of these would be sufficient). In general, it would be an empirical matter whether convexity applied. The point needs to be made though, that the convexity or otherwise of conditional market rent functions (such as the f and g conditional functions which are used here for illustration) depend upon the characteristics of the "parent" optimal value functions from which they have been derived, as well as upon the characteristics of the distance parameter functions $\mathbf{a}(r)$.

Another important point that needs to be made here is the continuity properties of the market rent function for the location good derived in this way. If there is complete non-coincidence of each locator's bid rent function, then the market rent function will consist of a locus of points which is the highest bid of a locator at that point and optimal locations for each locator will be unique. This case is clearly envisaged by Alonso (1960) and (1964) in his residential model, where market rent is a smooth continuous function of distance. Alternatively, if the market rent function is supported from below by families of bid rent functions, where within classes of locators there is complete coincidence of bid rent, and between classes of locators there is complete non-coincidence of bid rent, then the market rent function will be a piecewise smooth continuous function. This is

[1] Wherever upper case F appears in plain print it refers to the general notion of "F convexity" and should not be confused with italicised upper case *F*, which always refers to the aspatial optimal value function of problem (1).

normally the case with the Von Thünen agricultural model where a class of locators will be a specific agricultural product type (Livesey, 1984). This is echoed in Fujita (1989), who elegantly derives a piecewise smooth continuous residential land market rent function with homogeneity within residential classes and complete heterogeneity between residential classes, where class differentiation could be determined by income, household size, tastes etc.. Under these conditions whilst an optimal location might exist for a locator, that optimal location will not be unique, but will be part of an interval of optimal radial locations which will form concentric neighbourhoods in location.

5. OPTIMAL LOCATION.

If one were to continue in the tradition of the Von Thünen problem, the next logical step, having established the characteristics of the market rent function $w(r)$, would be to substitute the market rent function into the optimal value functions for the inner problems of (1) and (2) to give a clear specification of the outer problems:

$$F_r(\mathbf{b}) \equiv \max_r \left\{ F\left(\mathbf{a}(r), \mathbf{b}, w(r)\right)\right\} \text{ ; and} \tag{9}$$

$$G_r(\mathbf{b}) \equiv \min_r \left\{ G\left(\mathbf{a}(r), \mathbf{b}, w(r)\right)\right\} . \tag{10}$$

The existence of a solution to (9) or (10) turns on the fact that one must establish that the functions F or G are continuous in r and that the domains of F or G are compact sets of r. The former is determined by the continuity of F or G in $\mathbf{a}$ and w and the continuity of $\mathbf{a}$ and w in r. The latter is determined by two boundary conditions. Firstly, the inner boundary is guaranteed by the fact that r must be non-negative so that the inner boundary is r=0. The outer boundary from the central place is more problematical. One has to appeal to the idea that there must be some finite number that marks the outer limit to location, $r = \bar{r}$, that is, radial distance cannot go on forever. Assuming differentiability and interior solutions, the first order conditions to (9) and (10) are:

$$\frac{\partial F}{\partial \mathbf{a}'}\frac{d\mathbf{a}}{dr} = -\frac{\partial F}{\partial w}\frac{dw}{dr} \text{ ; and} \tag{11}$$

$$\frac{\partial G}{\partial \mathbf{a}'}\frac{d\mathbf{a}}{dr} = -\frac{\partial G}{\partial w}\frac{dw}{dr} . \tag{12}$$

These conditions both indicate that optimal location relative to the central place is determined by a balance between centripetal forces (left hand side of (11) and (12)) and the centrifugal force of location good price (right hand side of (11) and (12)). Equations (11) and (12) are general cases of the so called access space tradeoff forces in location, and through the use of Hotelling's lemma or Roy's identity and Shephard's lemma, generalise the classic Muth equation of the urban residential model (Muth, 1971; p.23, equation (3') and Turnbull, 1995, p.10, equation (2.5)).

6. DUAL PROBLEMS IN OPTIMAL LOCATION.

An interesting alternative to this approach would be to exploit further the insights of duality theory by considering the choice of r under conditions which are logically equivalent but which require consideration of alternative optimisations to the outer problems (9) and (10). Consider, for example, dual problems to (9) or (10), which may be represented as:

$$\underline{R}(\mathbf{b},f) \equiv \min_{r}\left\{ r \in \Re_{+} : \omega\left(\mathbf{a}(r),\mathbf{b},f\right) - w(r) \geq 0\right\} \; ; \tag{13}$$

$$\overline{R}(\mathbf{b},f) \equiv \max_{r}\left\{ r \in \Re_{+} : \omega\left(\mathbf{a}(r),\mathbf{b},f\right) - w(r) \geq 0\right\} \; : \text{and} \tag{14}$$

$$\underline{R^{G}}(\mathbf{b},g) \equiv \min_{r}\left\{ r \in \Re_{+} : \omega^{G}\left(\mathbf{a}(r),\mathbf{b},g\right) - w(r) \geq 0\right\} \; ; \tag{15}$$

$$\overline{R^{G}}(\mathbf{b},g) \equiv \max_{r}\left\{ r \in \Re_{+} : \omega^{G}\left(\mathbf{a}(r),\mathbf{b},g\right) - w(r) \geq 0\right\} \; . \tag{16}$$

These give the closest or furthest distance, r, from the central focal point of the Von Thünen system that the entity can locate. The optimal value functions of problems (13) to (16) are useful in location problems where optimal locations form an interval of radial distances, which are the classic concentric zones of the circular Von Thünen type model. Comparative statics on the radial extent of these concentric zones can be undertaken through the envelope properties of the optimal value functions of (13) to (16).

Figure 1 illustrates problems (13) and (14) for a homogeneous class of locators: such as the producers of a specific crop in the traditional agrarian problem; a class of households (say of the same income and tastes), as in the urban residential problem; or a specific industry for urban industrial location.

In Figure 1, $\omega(f,r)$ is a bid rent function for the homogeneous class of locators and that group face a market rent function $w(r)$. The shading in the figure shows the feasible set of bids for the problems in (13) and (14). That is, the set of distance/bid price co-ordinates where the difference between bid rent and market rent is non-negative. The solution to problem (13) is r^1 and that for (14) is r^2.

The optimal value functions of problems (13) to (16) are spatial functions because r has been optimised. The envelope properties of these functions can be explored by utilising the envelope properties of previous functions. For example, this can be done by forming an identity by substituting the aspatial function F from (9) into (13):

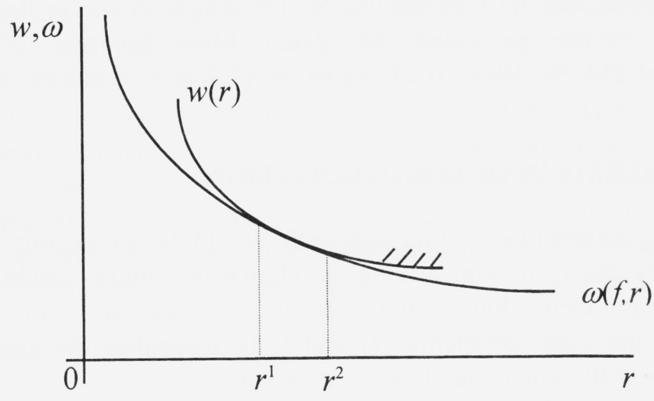

Figure 1. The Optimal Set of Locations in Minimising Rents.

$$\underline{R}\big(\mathbf{b}, F(\mathbf{a}(r), \mathbf{b}, w(r))\big) \equiv r \ . \tag{17}$$

To interpret this identity, note that the aspatial function F contains the market rent function as its argument. The optimal distance is found where bid rent equals market rent, and the $\underline{R}$ function conditioned on optimal bid rent (for a given r) clearly returns the same r as optimal.

Differentiating this identity with respect to r gives:

$$\frac{\partial \underline{R}}{\partial f} = \frac{1}{\dfrac{\partial F}{\partial \mathbf{a}'}\dfrac{d\mathbf{a}}{dr} + \dfrac{\partial F}{\partial w}\dfrac{dw}{dr}} \ ; \tag{18}$$

where the denominator of the right hand side indicates the fundamental centripetal and centrifugal forces of location in the Von Thünen location model. Differentiating the same identity with respect to $\mathbf{b}$ gives:

$$\frac{\partial \underline{R}}{\partial \mathbf{b}} = -\frac{\partial \underline{R}}{\partial f}\frac{\partial F}{\partial \mathbf{b}} \ ; \text{and} \tag{19}$$

utilising (18) and (19), gives:

$$\frac{\partial \underline{R}}{\partial \mathbf{b}} = -\frac{\dfrac{\partial F}{\partial \mathbf{b}}}{\dfrac{\partial F}{\partial \mathbf{a}'}\dfrac{d\mathbf{a}}{dr} + \dfrac{\partial F}{\partial w}\dfrac{dw}{dr}} \ .$$

The gradient vector $\partial F/\partial \mathbf{b}$ on the right hand side is known from the envelope properties of the function F. The sign of (18) is crucial in determining locational comparative statics. Given the nature of problems (13) to (16) then:

Proposition 2

When not equal to zero, $\partial \underline{R}/\partial f$ (or $\partial \underline{R^G}/\partial g$) is of opposite sign to $\partial \overline{R}/\partial f$ (or $\partial \overline{R^G}/\partial g$).

However, it is possible to go further than this proposition. Recalling the first order conditions (11) or (12), then for small perturbations around optimal locations such as r^1 and r^2 of Figure 1 the sign of (18) will be known. In this way comparative statics utilising (19) are feasible.

7. THE DEMAND FOR THE LOCATION GOOD AND LOCATION.

Finally, the demand for the location good can be examined by retrieving the relevant demand, via the envelope theorem, from the aspatial functions F, G, ω or ω^G. These demand functions will be functions of radial distance r and their relationship with r can be examined by differentiation.

However, recall that demand for the location good retrieved from ω or ω^G are conditioned on levels in F or G and are abstracted from own-price. This will allow some decomposition into the various effects on demand over radial distance. For example, take the demand retrieved from the function ω and the function F. One element in the vector $\mathbf{x}$ will refer to location good. Call this x. Then the two relevant demand functions are:

$$x^w = x(\mathbf{a}(r), \mathbf{b}, w(r)) \text{ ; and} \tag{20}$$

$$x^f = x^f(\mathbf{a}(r), \mathbf{b}, f) \text{ .} \tag{21}$$

The function (20) is the unconditional demand from the function F and is a function of own-price, hence the superscript w. The function (21) is the conditional demand from $\omega(\mathbf{a}(r), \mathbf{b}, f)$ and because it is conditioned on f it has been given the superscript f.

Differentiating (20) with respect to r gives:

$$\frac{\partial x^w}{\partial r} = \frac{\partial x^w}{\partial \mathbf{a}'}\frac{d\mathbf{a}}{dr} + \frac{\partial x^w}{\partial w}\frac{dw}{dr} \text{ .} \tag{22}$$

Alternatively substituting the aspatial function F from (9) into (21) gives the identity:

$$x^w \equiv x^f\left(\mathbf{a}(r), \mathbf{b}, F(\mathbf{a}(r), \mathbf{b}, w(r))\right) \text{ .} \tag{23}$$

Differentiating (23) with respect to r gives:

$$\frac{\partial x^w}{\partial r}=\frac{\partial x^f}{\partial \mathbf{a}'}\frac{d\mathbf{a}}{dr}+\frac{\partial x^f}{\partial f}\frac{\partial F}{\partial \mathbf{a}'}\frac{d\mathbf{a}}{dr}+\frac{\partial x^f}{\partial f}\frac{\partial F}{\partial w}\frac{dw}{dr}\ . \tag{24}$$

Comparison of (22) and (24) suggests the following decomposition:

$$\frac{\partial x^w}{\partial \mathbf{a}}=\frac{\partial x^f}{\partial \mathbf{a}}+\frac{\partial x^f}{\partial f}\frac{\partial F}{\partial \mathbf{a}};\ \text{and}\ \ \frac{\partial x^w}{\partial w}=\frac{\partial x^f}{\partial f}\frac{\partial F}{\partial w}\ ;$$

so that ambiguity in the term $\partial x^w/\partial \mathbf{a}$, if it arises, may be explained in terms of the components made up of the conditional effect of **a**, a substitution like effect and the "level" effect operating through F. Where effects remain ambiguous, there is the option to make use of the previously discussed relationships between F and dual functions such as R, in order to split up effects further into terms which may be separately capable of unambiguous interpretation. This approach allows a clear statement of what may or may not be deduced from theory and what must be left to empirical examination in terms of the demand for a location fixed good and location in a Von Thünen type model.

8. SUMMARY.

Figure 2 summarises the application of duality to a generalised Von Thünen model.

Starting with the central box it is possible to retrieve demand functions for the location good from the aspatial functions F or G. This is the upper central box in Figure 2. Alternatively, if the structure of $w(r)$ is known, choosing r to maximise or minimise the relevant function F or G gives the spatial optimal value functions F_r or G_r, the box on the left in the figure. The bid price functions ω or ω^G, conditioned on levels f or g, can be found by inversion of the aspatial functions F or G. Conditional demand functions, conditioned on f or g, can be retrieved from these functions. Alternatively, these maximum value functions can be obtained from the problem of maximising offer price for the locational good conditioned on levels in f or g. The aspatial functions F and G can then be retrieved by inversion.

The functions ω or ω^G can be used to specify problems dual to the maximum or minimum problems in F or G. These are choose r to minimise or maximise radial distance from central place, subject to a market price constraint. The optimal value functions to these problems are in the lowest central box in Figure 2. Substitution of F or G into $\underline{R}$ and $\overline{R}$ or $\underline{R}^G$ and $\overline{R}^G$ gives the minimum ($\underline{r}$) and maximum ($\overline{r}$) distances respectively. The envelope properties of these location functions, R, form a useful route to location comparative statics. Conversely, substitution of these maximum or minimum location functions, R, into the relevant aspatial functions F or G gives levels in these functions at the optimal location, that is F_r or G_r. Finally, substitution of the relevant optimal value function R into the offer price function ω or ω^G retrieves prices for the location good at maximum or minimum distance conditioned on levels in f or g.

This paper has outlined an application of duality theory to a general location model. The result has been the development of dual functions beyond standard duality theory with which to analyse location issues. The model is general enough to be applied to problems of urban business location as well as urban residential location and the fundamental rural problem of Von Thünen. Future development of the model could take place to capture specific models of regional location in production, for example, an extension of the model to include centralising ("backwash effects") and decentralising forces ("spread effects") between regions of Hirschman (1988) and the regional centre/periphery model of Friedmann (1972). A difficulty here is the specification of the decentralising or "spread" forces. In the present general model these have been encapsulated through the effect of competition for central location on prices of the location good. However, the "spread effects" of regional models appear to be wider than this, including the increasing real wages of central regions and the effect of congestion on production and distribution cost, as well as the "spread" of economic development to peripheral regions through increased trade with fast growing central regions. This would require a clear development of the structure of these decentralising forces across distance from central region and an incorporation of that structure into the objective functions of and the constraints faced by producers in regional location.

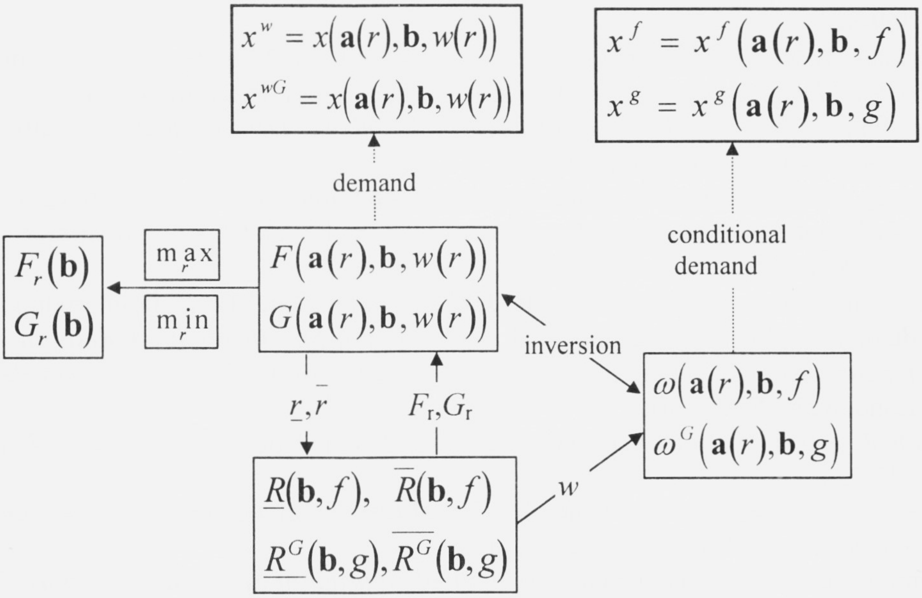

Figure 2. The Dual Relations in the Generalised Von Thünen Optimal Location Problem.

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